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On general balance of designs for two-way elimination of heterogeneity

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SUMMARY

In this paper we consider the general balance of designs for two-way elimination of heterogeneity. In particular cases, the property is discussed in relation to strictly orthogonal and efficiency balanced designs.

KEY WORDS: design for two-way elimination of heterogeneity, general balance, orthogonal block structure, commutativity, strict orthogonality, efficiency balance.

1. Introduction

Let us consider a design for two-way elimination of heterogeneity in which v treatments are allocated to n experimental units arranged in b_1 rows and b_2 columns. The appropriate mixed model may be expressed as

$$\mathbf{y} = \mathbf{\Delta}'\boldsymbol{\tau} + \mathbf{D}_1'\boldsymbol{\beta}_1 + \mathbf{D}_2'\boldsymbol{\beta}_2 + \mathbf{u} \tag{1}$$

where $\mathbf{y}$ is an $n \times 1$ vector of observations, $\mathbf{\Delta}'$, $\mathbf{D}_1'$ and $\mathbf{D}_2'$ - are $n \times v$, $n \times b_1$, $n \times b_2$ design matrices for treatments, rows and columns, respectively, $\boldsymbol{\tau}$, $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ - are $v \times 1$, $b_1 \times 1$, and $b_2 \times 1$ vectors of unknown treatment effects, of unknown row effects and of unknown column effects, respectively, and $\mathbf{u}$ is an $n \times 1$ vector of errors. Let $\mathbf{1}_a$ denote the $a \times 1$ vector of ones. Further, let $\mathbf{r} = \mathbf{\Delta}\mathbf{1}_n$, $\mathbf{k}_1 = \mathbf{D}_1\mathbf{1}_n$, and $\mathbf{k}_2 = \mathbf{D}_2\mathbf{1}_n$ denote the vector of treatment replications, the vector of row sizes, and the vector of column sizes, respectively, and let $\mathbf{R} = \mathbf{\Delta}\mathbf{\Delta}'$, $\mathbf{K}_1 = \mathbf{D}_1\mathbf{D}_1'$, and $\mathbf{K}_2 = \mathbf{D}_2\mathbf{D}_2'$ denote the diagonal matrices with the elements of $\mathbf{r}$, $\mathbf{k}_1$, and $\mathbf{k}_2$ on their diagonals. Moreover, let $\mathbf{N}_1 = \mathbf{\Delta}\mathbf{D}_1'$ be the $(v \times b_1)$ treatment-row incidence matrix, let $\mathbf{N}_2 = \mathbf{\Delta}\mathbf{D}_2'$ be the $(v \times b_2)$ treatment-column incidence matrix, and let $\mathbf{N}_{12} = \mathbf{D}_1\mathbf{D}_2'$ be the $(b_1 \times b_2)$ row-column incidence matrix.

2. Orthogonal block structure

In model (1) the components β_{1_i}, β_{2_j}, and u_{ijk} of $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$, and $\mathbf{u}$ are uncorrelated random variables with zero means and variances

$$\mathrm{Var}(\beta_{1_i}) = \sigma^2_{\beta_1}, \mathrm{Var}(\beta_{2_j}) = \sigma^2_{\beta_2}, \text{ and } \mathrm{Var}(u_{ijk}) = \sigma^2. \tag{2}$$

The dispersion matrices are

$$\mathrm{D}(\boldsymbol{\beta}_1) = \sigma^2_{\beta_1}\mathbf{I}_{\beta_1}, \mathrm{D}(\boldsymbol{\beta}_2) = \sigma^2_{\beta_2}\mathbf{I}_{\beta_2}, \text{ and } \mathrm{D}(\mathbf{u}) = \sigma^2\mathbf{I}_n, \tag{3}$$

where $\mathbf{I}_a$ denotes the identity matrix of order a. Because treatment parameters τ_k in $\boldsymbol{\tau}$ are constant, we have

$$\mathrm{Var}(y_{ij}) = \sigma^2 + \sigma^2_{\beta_1} + \sigma^2_{\beta_2}$$

and

$$\mathrm{Cov}(y_{ij}, y_{i'j'}) = \begin{cases} \sigma^2_{\beta_1} & i = i' \quad j \neq j' \\ \sigma^2_{\beta_2} & i \neq i' \quad j = j' \\ 0 & i \neq i' \quad j \neq j'. \end{cases}$$

Then the expected value and the dispersion matrix for the observations are

$$\mathrm{E}(\mathbf{y}) = \boldsymbol{\Delta}'\boldsymbol{\tau}$$

and

$$\mathrm{D}(\mathbf{y}) = \mathbf{V} = \sigma^2_{\beta_1}\mathbf{D}_1'\mathbf{D}_1 + \sigma^2_{\beta_2}\mathbf{D}_2'\mathbf{D}_2 + \sigma^2\mathbf{I}_n.$$

We first consider the orthogonal blocking structure of the design.

DEFINITION 1. (Houtman and Speed, 1983). A design is said to have an orthogonal block structure if the $n \times n$ dispersion matrix $\mathbf{V}$ can be written as

$$\mathbf{V} = \sum_{p=0}^{t} \sigma_p \mathbf{S}_p,$$

where $\sigma_p \geq 0$ for all p, and $\mathbf{S}_p$ are symmetric, idempotent, pairwise orthogonal matrices summing up to the identity matrix, i.e.

$$\mathbf{S}_p = \mathbf{S}_p' = \mathbf{S}_p^2, \; \mathbf{S}_p\mathbf{S}_q = \mathbf{S}_q\mathbf{S}_p = 0 \text{ if } p \neq q, \text{ and } \sum_p \mathbf{S}_p = \mathbf{I}_n. \tag{4}$$

Let

$$\mathbf{S}_0 = \mathbf{1}_n\mathbf{1}_n'/n,\ \mathbf{S}_1 = \mathbf{D}_1'\mathbf{K}_1{}^{-1}\mathbf{D}_1 - \mathbf{S}_0,\ \mathbf{S}_2 = \mathbf{D}_2'\mathbf{K}_2{}^{-1}\mathbf{D}_2 - \mathbf{S}_0,$$
and
$$\mathbf{S}_3 = \mathbf{I} - \mathbf{D}_1'\mathbf{K}_1{}^{-1}\mathbf{D}_1 - \mathbf{D}_2'\mathbf{K}_2{}^{-1}\mathbf{D}_2 + \mathbf{S}_0. \tag{5}$$

Then

$$\mathbf{V} = \sigma_0^2\mathbf{S}_0 + \sigma_1^2\mathbf{S}_1 + \sigma_2^2\mathbf{S}_2 + \sigma_3^2\mathbf{S}_3,$$

where $\sigma_0^2 = \sigma_1^2 + \sigma_2^2 - \sigma_3^2 = b_2\sigma_{\beta_1}^2 + b_1\sigma_{\beta_2}^2 + \sigma^2$, $\sigma_1^2 = b_2\sigma_{\beta_1}^2 + \sigma^2$, $\sigma_2^2 = b_1\sigma_{\beta_2}^2 + \sigma^2$, and $\sigma_3^2 = \sigma^2$.

For matrices $\mathbf{S}_i$, $i = 0, 1, 2$, and 3, which are defined in (5), the conditions (4) do not always hold. Thereby, in general, for the conditions (5), a design for two-way elimination of heterogeneity does not necessarily has the orthogonal block structure. However, conditions (4) are satisfied for designs with the matrix $\mathbf{N}_{12} = \mathbf{1}_{b_1}\mathbf{1}_{b_2}'$, which are usually called row-column designs. These designs were considered by Houtman and Speed (1983) and Mejza (1992). More general designs for two-way elimination of heterogeneity, i.e. designs with $\mathbf{N}_{12} = \mathbf{k}_1\mathbf{k}_2'/n$, have the orthogonal block structure, also.

Now, we can formulate

THEOREM 1. *A design for two-way elimination of heterogeneity corresponding to the linear model (1) with (2), (3), and (4) is a design with the orthogonal block structure.*

3. Commutativity

DEFINITION 2. A design for two-way elimination of heterogeneity is said to have the commutativity property if $\mathbf{N}_1\mathbf{K}_1^{-1}\mathbf{N}_1'\mathbf{R}^{-1}\mathbf{N}_2\mathbf{K}_2^{-1}\mathbf{N}_2' = \mathbf{N}_2\mathbf{K}_2^{-1}\mathbf{N}_2'\mathbf{R}^{-1}\mathbf{N}_1\mathbf{K}_1^{-1}\mathbf{N}_1'$.

We will denote

$$\mathbf{C}_0 = \mathbf{R} - \mathbf{r}\mathbf{r}'/n,\ \mathbf{C}_1 = \mathbf{R} - \mathbf{N}_1\mathbf{K}_1^{-1}\mathbf{N}_1',\ \mathbf{C}_2 = \mathbf{R} - \mathbf{N}_2\mathbf{K}_2^{-1}\mathbf{N}_2',$$

$$\mathbf{C}_r = \mathbf{N}_1\mathbf{K}_1^{-1}\mathbf{N}_1' - \mathbf{r}\mathbf{r}'/n,\ \mathbf{C}_c = \mathbf{N}_2\mathbf{K}_2^{-1}\mathbf{N}_2' - \mathbf{r}\mathbf{r}'/n,\ \text{and}$$

$$\mathbf{C}_{rc} = \mathbf{R} - \mathbf{N}_1\mathbf{K}_1^{-1}\mathbf{N}_1' + \mathbf{N}_2\mathbf{K}_2^{-1}\mathbf{N}_2' + \mathbf{r}\mathbf{r}'/n.$$

Moreover,

$$\mathbf{A}_1 = \mathbf{R}^{-1/2}\mathbf{C}_1\mathbf{R}^{-1/2},\ \mathbf{A}_2 = \mathbf{R}^{-1/2}\mathbf{C}_2\mathbf{R}^{-1/2},$$

$$\mathbf{A}_r = \mathbf{R}^{-1/2}\mathbf{C}_r\mathbf{R}^{-1/2},\ \mathbf{A}_c = \mathbf{R}^{-1/2}\mathbf{C}_c\mathbf{R}^{-1/2},\ \text{and}\ \mathbf{A}_{rc} = \mathbf{R}^{-1/2}\mathbf{C}_{rc}\mathbf{R}^{-1/2}.$$

The following lemma is concerned with relationships between the property of commutativity and the matrices defined above.

LEMMA 1. *For a design for two-way elimination of heterogeneity corresponding to the linear model (1), the following statements are equivalent:*

a) $\mathbf{N}_1\mathbf{K}_1^{-1}\mathbf{N}_1'\mathbf{R}^{-1}\mathbf{N}_2\mathbf{K}_2^{-1}\mathbf{N}_2' = \mathbf{N}_2\mathbf{K}_2^{-1}\mathbf{N}_2'\mathbf{R}^{-1}\mathbf{N}_1\mathbf{K}_1^{-1}\mathbf{N}_1'$,

b) $\mathbf{C}_1\mathbf{R}^{-1}\mathbf{C}_2 = \mathbf{C}_2\mathbf{R}^{-1}\mathbf{C}_1$,

c) $\mathbf{A}_1\mathbf{A}_2 = \mathbf{A}_2\mathbf{A}_1$,

d) $\mathbf{A}_r\mathbf{A}_c = \mathbf{A}_c\mathbf{A}_r$,

e) $\mathbf{C}_r\mathbf{R}^{-1}\mathbf{C}_c = \mathbf{C}_c\mathbf{R}^{-1}\mathbf{C}_r$.

4. General balance

Now, let $\mathbf{T} = \mathbf{\Delta}'(\mathbf{\Delta}\mathbf{\Delta}')^{-1}\mathbf{\Delta} = \mathbf{\Delta}'\mathbf{R}^{-1}\mathbf{\Delta}$. We notice that $\mathbf{T}^2 = \mathbf{T}$.

DEFINITION 3. (Speed, 1983). A design with an orthogonal block structure is said to have the general balance property if the matrices $\mathbf{TS}_0\mathbf{T}$, $\mathbf{TS}_1\mathbf{T}$, ..., $\mathbf{TS}_t\mathbf{T}$ mutually commute, i.e. $\mathbf{TS}_i\mathbf{TS}_j\mathbf{T} = \mathbf{TS}_j\mathbf{TS}_i\mathbf{T}$, $i \neq j = 0, 1, \dots, t$.

It is obvious that for a design for two-way elimination of heterogeneity satisfying (5), the matrix $\mathbf{TS}_0\mathbf{T}$ commutes with the matrices $\mathbf{TS}_i\mathbf{T}$, $i = 1, 2, 3$. On the other hand, the commutativity property of matrices $\mathbf{TS}_i\mathbf{T}$ is equivalent to commutativity of matrices $\mathbf{R}^{-1/2}\mathbf{\Delta}\mathbf{S}_i\mathbf{\Delta}'\mathbf{R}^{-1/2}$. Since $\mathbf{\Delta}\mathbf{S}_1\mathbf{\Delta}' = \mathbf{C}_r$, $\mathbf{\Delta}\mathbf{S}_2\mathbf{\Delta}' = \mathbf{C}_c$, and $\mathbf{\Delta}\mathbf{S}_3\mathbf{\Delta}' = \mathbf{C}_{rc}$, then the general balance is satisfied if and only if the matrices $\mathbf{A}_r$, $\mathbf{A}_c$, and $\mathbf{A}_{rc}$ all commute. Since $\mathbf{C}_{rc} = \mathbf{C}_0 - \mathbf{C}_r - \mathbf{C}_c$ we have

$$\mathbf{A}_r\mathbf{A}_c = \mathbf{A}_c\mathbf{A}_r \ \text{ iff } \ \mathbf{C}_r\mathbf{R}^{-1}\mathbf{C}_c = \mathbf{C}_c\mathbf{R}^{-1}\mathbf{C}_r,$$

$$\mathbf{A}_r\mathbf{A}_{rc} = \mathbf{A}_{rc}\mathbf{A}_r \ \text{ iff } \ \mathbf{C}_r\mathbf{R}^{-1}\mathbf{C}_c = \mathbf{C}_c\mathbf{R}^{-1}\mathbf{C}_r,$$

$$\mathbf{A}_c\mathbf{A}_{rc} = \mathbf{A}_{rc}\mathbf{A}_c \ \text{ iff } \ \mathbf{C}_r\mathbf{R}^{-1}\mathbf{C}_c = \mathbf{C}_c\mathbf{R}^{-1}\mathbf{C}_r,$$

Now, using Lemma 1 we have

THEOREM 2. *A design for two-way elimination of heterogeneity with the orthogonal block structure corresponding to the linear model (1) with (2) and (3) is generally balanced if and only if the design has the commutativity property.*

Eccleston and Russell (1975) introduced the concept of orthogonality. For further discussion see also Eccleston and Russell (1977) and Siatkowski (1993).

DEFINITION 4. A design for two-way elimination of heterogeneity is said to have its rows and columns strictly orthogonal if

$$\mathbf{N}_1'\mathbf{R}^{-1}\mathbf{N}_2 = \mathbf{N}_{12}.$$

The next theorem deals with a relationship between the property of strict orthogonality and the property of general balance.

THEOREM 3. *If a design for two-way elimination of heterogeneity corresponding to the linear model (1) with (2) and (3) is such that rank*$(\mathbf{N}_{12}) = 1$ *and has its rows and columns strictly orthogonal, then the design is generally balanced.*

Proof. It is known that $\text{rank}(\mathbf{N}_{12}) = 1$ is equivalent to $\mathbf{N}_{12} = \mathbf{k}_1\mathbf{k}_2'/n$. Hence

$$\mathbf{N}_1\mathbf{K}_1^{-1}\mathbf{N}_1'\mathbf{R}^{-1}\mathbf{N}_2\mathbf{K}_2^{-1}\mathbf{N}_2' = \mathbf{N}_1\mathbf{K}_1^{-1}\mathbf{N}_{12}\mathbf{K}_2^{-1}\mathbf{N}_2' = \mathbf{r}\mathbf{r}'/n$$

and

$$\mathbf{N}_2\mathbf{K}_2^{-1}\mathbf{N}_2'\mathbf{R}^{-1}\mathbf{N}_1\mathbf{K}_1^{-1}\mathbf{N}_1' = \mathbf{N}_2\mathbf{K}_2^{-1}\mathbf{N}_{12}'\mathbf{K}_1^{-1}\mathbf{N}_1' = \mathbf{r}\mathbf{r}'/n$$

imply the commutativity property. According to Theorem 2, the design is generally balanced. □

Theorem 3 may be viewed as an extended version of the implication given in the section 6 of Mejza (1992), who considered designs with $\mathbf{N}_{12} = \mathbf{1}_n\mathbf{1}_n'$, only.

LEMMA 2. *For a design for two-way elimination of heterogeneity corresponding to the linear model (1), the following statements are equivalent:*

a) rank$(\mathbf{N}_h) = 1$,

b) $\mathbf{N}_h = \mathbf{r}\mathbf{k}_h'/n$,

c) $\mathbf{N}_h\mathbf{K}_h^{-1}\mathbf{N}_h' = \mathbf{r}\mathbf{r}'/n$

where $h = 1$ *or* 2.

THEOREM 4. *A design for two-way elimination of heterogeneity with the orthogonal block structure corresponding to the linear model (1) satisfying one of the properties (a) – (c) of Lemma 2 is generally balanced.*

Berube and Styan (1993) and Siatkowski (1994) considered the class of designs for two-way elimination of heterogeneity satisfying $\mathbf{C} = \xi_1\mathbf{C}_1 + \xi_2\mathbf{C}_2 - \xi_0\mathbf{C}_0$, where ξ_1, ξ_2, and $\xi_0 > 0$. Now, we show that for this class, the general balance property is implied by the efficiency balance property, i.e. $\mathbf{C} = \theta(\mathbf{R} - \mathbf{r}\mathbf{r}'/n)$ for some $\theta \in (0,1]$ or by efficiency-balance of the treatment-row subdesign, i.e. $\mathbf{C}_1 = \theta_1(\mathbf{R} - \mathbf{r}\mathbf{r}'/n)$ for some $\theta_1 \in (0,1]$ or by efficiency-balance of the treatment-column subdesign, i.e. $\mathbf{C}_2 = \theta_2(\mathbf{R} - \mathbf{r}\mathbf{r}'/n)$ for some $\theta_2 \in (0, 1]$.

THEOREM 5. *If a connected design for two-way elimination of heterogeneity with the orthogonal block structure satisfying* $\mathbf{C} = \xi_1\mathbf{C}_1 + \xi_2\mathbf{C}_2 - \xi_0\mathbf{C}_0$ *is efficiency-balanced or if its treatment-row or its treatment-column subdesign is efficiency-balanced, then the design is generally balanced.*

Proof. From Theorem 4 (Berube and Styan, 1993), if assumptions of the theorem hold, it follows that the design satisfies the commutativity property, and hence, in view of Theorem 2, the design is generally balanced. □

5. Example

Speed (1983) claims that "all row-column designs, i.e. designs for two-way elimination of heterogeneity with $\mathbf{N}_{12} = \mathbf{1}_n\mathbf{1}_n'$, that have ever been used in practice satisfy the property of general balance". However, designs that do not satisfy this condition exist. A simple example is due to Speed (1983). Another example is given below.

Example. For the equireplicated row-column design

$$\begin{matrix} 1 & 4 & 2 & 5 & 3 \\ 4 & 3 & 5 & 1 & 5 \\ 1 & 2 & 3 & 1 & 2 \\ 5 & 4 & 3 & 2 & 4 \\ 3 & 2 & 5 & 4 & 1 \end{matrix}$$

we have

$$\mathbf{C}_r = \frac{1}{5}\begin{pmatrix} 2 & 1 & 0 & -2 & -1 \\ 1 & 2 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & 2 & 1 \\ -1 & -2 & 0 & 1 & 2 \end{pmatrix}, \quad \mathbf{C}_c = \frac{1}{5}\begin{pmatrix} 4 & -2 & -2 & 0 & 0 \\ -2 & 2 & 0 & 1 & -1 \\ -2 & 0 & 2 & -1 & 1 \\ 0 & 1 & -1 & 2 & -2 \\ 0 & -1 & 1 & -2 & 2 \end{pmatrix},$$

$$\mathbf{C}_r\mathbf{R}^{-1}\mathbf{C}_c = \frac{1}{125}\begin{pmatrix} 6 & -3 & -3 & -1 & 1 \\ 0 & 3 & -3 & 4 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ -6 & 3 & 3 & 1 & -1 \\ 0 & -3 & 3 & -4 & 4 \end{pmatrix},$$

$$\mathbf{C}_c\mathbf{R}^{-1}\mathbf{C}_r = \frac{1}{125}\begin{pmatrix} 6 & 0 & 0 & -6 & 0 \\ -3 & 3 & 0 & 3 & -3 \\ -3 & -3 & 0 & 3 & 3 \\ -1 & 4 & 0 & 1 & -4 \\ 1 & -4 & 0 & -1 & 4 \end{pmatrix},$$

and $\mathbf{C}_r\mathbf{R}^{-1}\mathbf{C}_c \neq \mathbf{C}_c\mathbf{R}^{-1}\mathbf{C}_r$, thus showing (in agreement with Lemma 1 and Theorem 2) that the property of general balance is not satisfied.

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